Example 9: Let $\mathbf{x}$ in $\mathbb{R}^n$. Show that $\mathbf{x}^T\mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \vec{x}^T\vec{x} = [x_1, x_2, \dots x_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1^2 + x_2^2 + \dots + x_n^2 \geq 0$$

only way to make $=0$ is if $\vec{x} = \vec{0}$

Example 10: Let A be a $m \times n$ matrix. Complete the steps to show that rank(A) = rank(A^TA).

This example proves theorem 3.28 (part a) in the Poole textbook.

1. Suppose $\mathbf{x}$ is in null(A). Show that $\mathbf{x}$ is in null(A^TA).

A^TA is a $n \times n$ matrix.

$\vec{x}$ in null(A), $A\vec{x} = \vec{0}$

$A^TA\vec{x} = A^T\vec{0} = \vec{0}$, $\vec{x}$ in null(A^TA)

A: $m \times n$ $\quad$ A^T: $n \times m$

A^TA: $(n \times m)(m \times n)$

2. Suppose $\mathbf{x}$ is in null(A^TA). Show that $\mathbf{x}$ is in null(A).

$\vec{x}$ in null(A^TA), $A^TA\vec{x} = \vec{0}$ $\qquad$ $(AB)^T = B^TA^T$

$\boxed{\text{Trick}}$ Calculate $(A\vec{x})^TA\vec{x} = \vec{x}^T A^TA\vec{x}$

$= \vec{x}^T\vec{0} = \vec{0}$

Thus $A\vec{x} = 0$ by example 9, and $\vec{x}$ in null(A)

3. Use null(A) = null(A^TA) and the rank-nullity theorem to show rank(A) = rank(A^TA).

rank(A) = n − nullity(A)

$= n$ − nullity(A^TA)

$=$ rank(A^TA)